Double Covering of Virtual Link and Pseudo Goeritz Matrix

G. Black, Y. Xuan

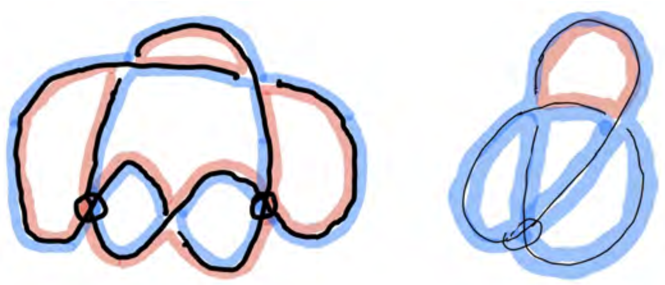
The Ohio State University

July 14, 2023

Checkerboard Coloring

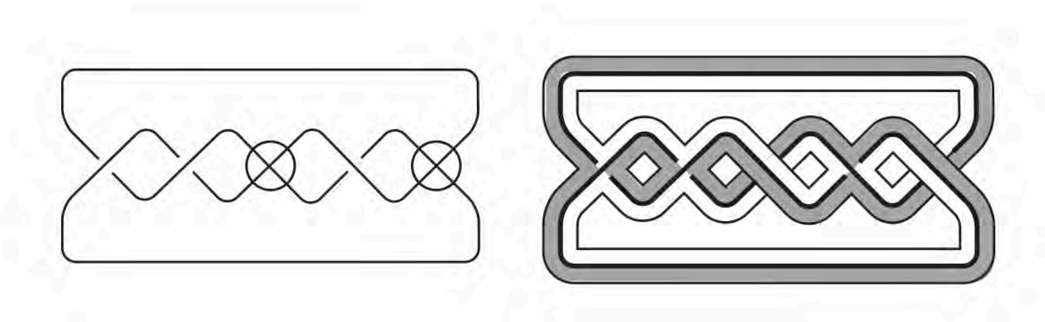
Let D be a virtual link diagram, a checkerboard coloring of D is a coloring of two sides of arcs such that

- no arc has two sides with the same color
- at each classical crossing the opposite regions have the same color
- at each virtual crossing the color doesn't change



Checkerboard Coloring

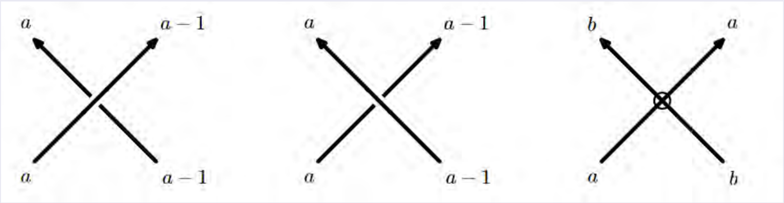
It is easier to see from the "abstract link diagrams"



Alexander Numbering

Definition 1

For an oriented virtual link diagram, an *Alexander numbering* is a coloring of its semi-arcs(arc between two classical crossings) given by the following relation:

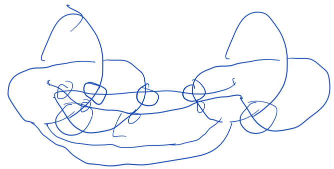


The checkerboard coloring of virtual link is equivalent to its Alexander numbering by $\mathbb{Z}/2\mathbb{Z}$.

Double Covering of Virtual Link Diagram

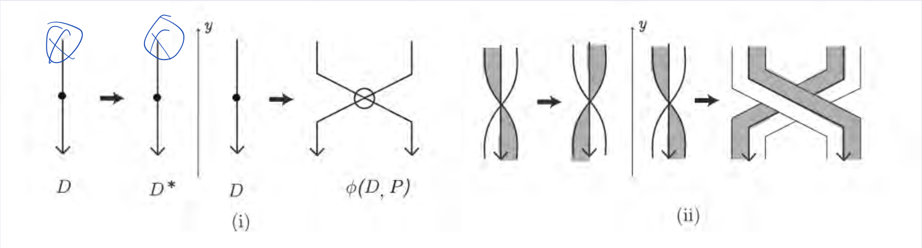
The checkerboard colorability is analogous to the orientation of manifolds. It is natural to think about if any virtual link has an orientable “double cover”. The answer is yes. In fact, A double cover of an oriented link diagram can be obtained through the following three ways (if not more):

Double Covering of Virtual Link Diagram

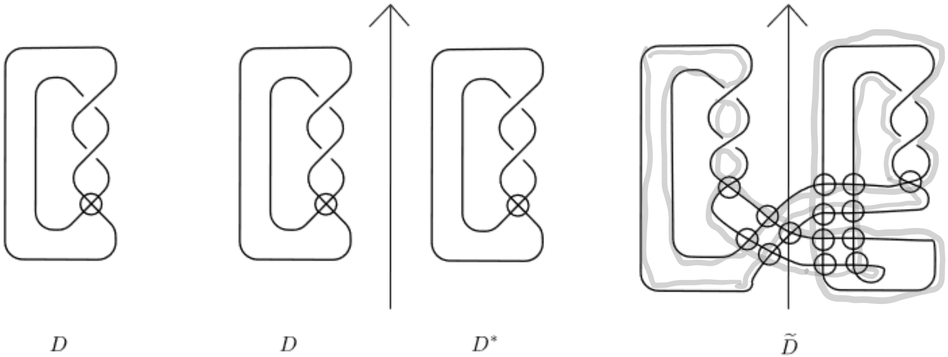


Method 1

Take two disjoint copies of D, at each virtual crossing, break the two leaving arcs and connect each to the other copy. If the arc has to cross other arcs, they cross by virtual crossings.

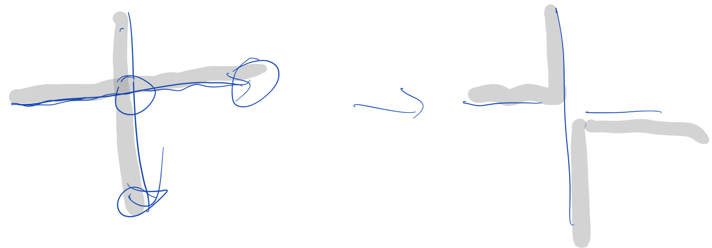


Double Covering of Virtual Link Diagram



(i) (ii) (iii)

Double Covering of Virtual Link Diagram

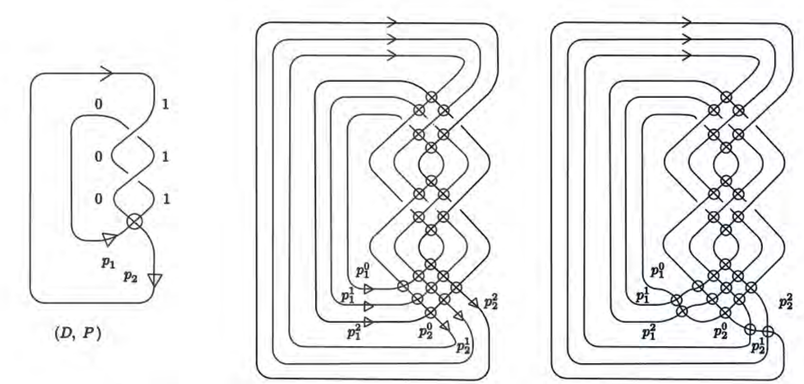


The idea of proof: if we replace all the virtual crossings by classical crossings, then the link will be checkerboard colorable. This is equivalent to twisting both strands at each virtual crossing. Take two copies of the original graph with the same orientaion and opposite checkerboard color, we can connect them and their colors are compatible.

Double Covering of Virtual Link Diagram

Remark

This method can be generalized to $\mathbb{Z}/n\mathbb{Z}$ Alexander numbering by connecting the cut points in a cyclic way. Notice that the copies cross each other by virtual crossings.



(i) (ii) (iii)

Double Covering of Virtual Link Diagram

Method 2

We replace every strand by two strands at each crossings in the following way:

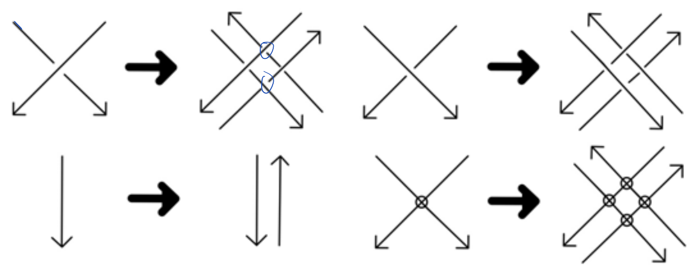


Fig. 9. A map φ_1.

Double Covering of Virtual Link Diagram

Method 3

Analogous to Method 2

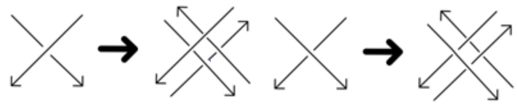


Fig. 11. A map φ_2.

Notice that the two copies are not connected and cross each other by a few classical crossings and virtual crossings.

Double Covering of Virtual Link Diagram

Method 2 and 3 provide checkerboard colorable diagrams:

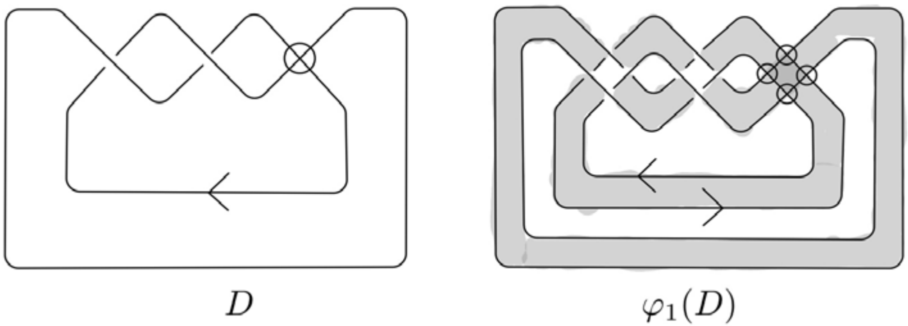


Fig. 10. An example of a virtual link diagram D and $\varphi_1(D)$.

Correspondence

Method 2 and 3 construct a 1-1 correspondence:

D	semi-arc	classical crossing	sign of corner
$\varphi_i(D)$	shaded region	unshaded square	sign of classical crossing

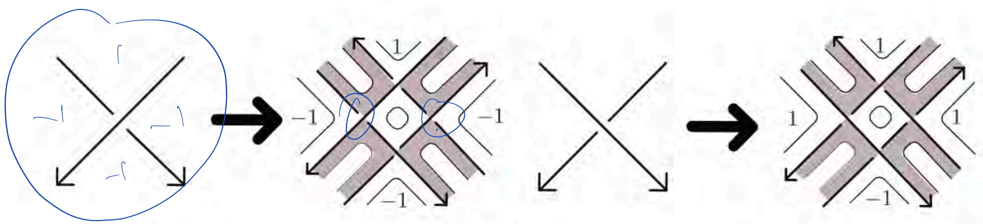


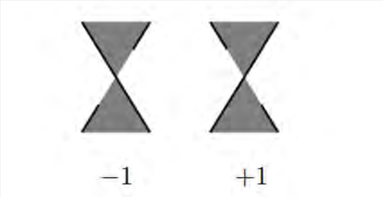
Fig. 18. A chessboard coloring of an ALD associated with $\varphi_1(D)$.

Goeritz Matrix

Recall the definition of Goeritz matrix:

Definition 2

The sign of a crossing $\eta(c)$:



Goeritz Matrix

Definition 3

Take a checkerboard colored link diagram D with the unbounded region shaded. Enumerate the shaded regions by $1, 2, \ldots, n$. The *pre-Goeritz matrix* of D is a $n \times n$ matrix (g_{ij}):

$$g_{ij} = \begin{cases} \sum_{\text{crossings between } i,j} \eta(c), & i \neq j \\ -\sum_{k \neq i} g_{ik}, & i = j \end{cases}$$

Remark

There are two pre-Goeritz matrices, corresponding to the unbounded region is shaded or unshaded. We can put the two matrices together in a larger matrix.

Warning

For unknown reason, we assume the links are connected, which means the components are linked by classical crossings only.

Pseudo Goeritz Matrix

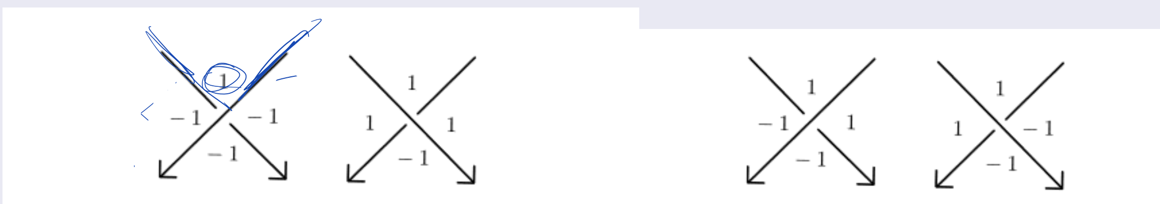
Definition 4

For a virtual diagram D, G_i, the pseudo Goeritz matrix of D is the Goeritz matrix of its double cover $\varphi_i(D)$.

Due to the previous correspondence, we know this is equivalent to another definition of pseudo-Goeritz matrix:

Definition 5

The first and second local index η:



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Pseudo Goeritz Matrix

Definition 6

For an oriented virtual diagram D with n semi-arcs(arcs between two classical crossings), the *pseudo Goeritz matrix* $G = (g_{ij})$ is an $n \times n$ matrix:

$$g_{ij} = \begin{cases} \sum_{\text{crossings between i,j}} \eta, & i \neq j \\ -\sum_{k \neq i} g_{ik}, & i = j \end{cases}$$

Torsion Invariant

A matrix A' is said to be obtained from a matrix A by an *elementary transformation* if A' is related with A as follows:

(i) A matrix A' is obtained from A by replacing the ith row (or the ith column) $\boldsymbol{a}_i$ of A with $-\boldsymbol{a}_i$.

(ii) A matrix A' is obtained from A by interchanging the ith row (or the ith column) and the jth row (or the jth column).

(iii) A matrix A' is obtained from A by adding the jth row (or the jth column) of A multiplied by an integer to the ith row (or the ith column).

(iv) A matrix A' is equal to $(A\mathbf{0})$.

(v) A matrix A' is equal to $A \oplus (1)$.

By applying a sequence of elementary transformations to a matrix G' obtained from a pre-Goeritz matrix $G(D)$ by deleting a row and a column, G' deforms to a diagonal integer matrix $(k_1) \oplus \ldots \oplus (k_d)$, where

- $k_1 \geqq 0$,
- $k_i \geqq 0$ and $k_i \neq 1$ for $1 \leqq i \leqq d (d \geqq 2)$ and k_{i+1} is an integer multiple of k_i if $1 \leqq i \leqq d-1 (d \geqq 2)$.

Torsion Invariant

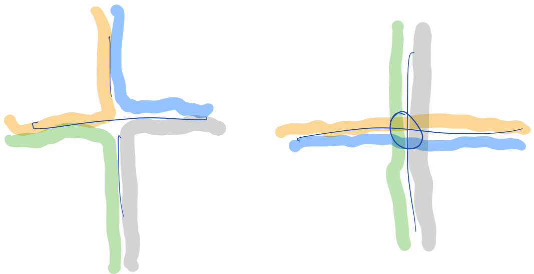
The sequence $(k_1, \ldots, k_d)$ is called torsion invariant of $G(D)$. It is known that for classical links, the torsion invariant is a link invariant

Theorem

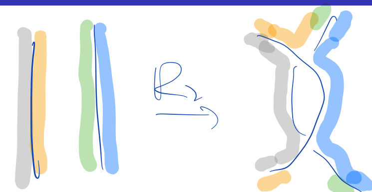
Torsion invariant of pseudo-Goeritz matrix is a virtual link invariant.

Connected Regions of Virtual Link

Before we color a virtual diagram, we must identify the regions to be colored.



Connected Regions of Virtual Link



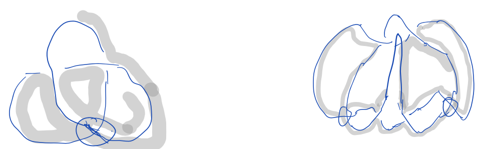
There are two reasons why a virtual diagram is not checkerboard colorable:

One is that there is a strand whose both sides are from the same region.

Example: 2.1

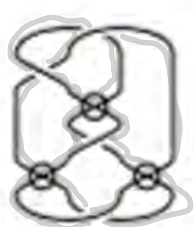
We say that a link diagram is separated if there is no strand whose both sides are from the same region. A link is separable if it admits a separated diagram.

Separated property is invariant under R and VR moves except for R_2 move.



Connected Regions of Virtual Link

Another reason is that the diagram is separated but the regions can't be checkerboard colored.
Example: 4.92.
Although we still have no checkerboard coloring to build a Goeritz matrix, can we use orientation and Kauffman state instead? Or can we build Dehn's coloring matrix?



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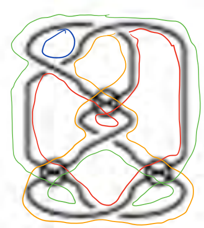

4.92

Dehn solvable ⟺ Checkerboard colorable?